Math 1320: Systems of Inequalities

What is a linear inequality? Previously, we worked with linear equations of the form $Ax + By = C$, whose graphs are straight lines. If we swap out the equals sign for $<$, $>$, $\leq$, or $\geq$, then we get a linear inequality, like the example below:

$$x + 2y > 3$$

A system of linear inequalities is when two or more inequalities are working together.

What are the solutions to a system of inequalities? Similar to solutions for systems of linear equations, we are looking for all ordered pairs (x, y) that make all inequalities in our system true. To solve a system of inequalities, we need to graph each inequality, then find the area that all graphs have in common (if there is one). This area is the solution set of all points that satisfy the inequalities.

How do we graph linear inequalities in two variables?

1. Replace the inequality symbol with an equal sign and graph the corresponding linear equation. Draw a solid line if the original inequality contains a $\leq$ or $\geq$ symbol. Draw a dashed line if the original inequality contains a $<$ or $>$ symbol.

2. Choose a test point from one of the half-planes. (Do not choose a point on the line.) Substitute the coordinates of the test point into the inequality.

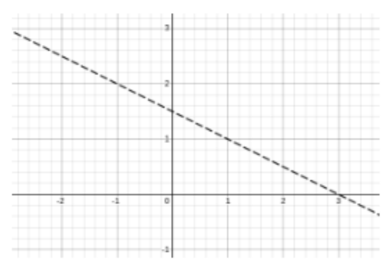
3. If a true statement results, shade the half-plane containing the test point. If a false statement results, shade the half-plane not containing the test point.

Example 1. Graphing a Linear Inequality in Two Variables

Graph: $x + 2y > 3$

Step 1: We need to graph $x + 2y = 3$. Let's find the x- and y-intercepts to graph the line.

Set $y = 0$ to find x-intercept	**Set $x = 0$ to find y-intercept**
$x + 2y = 3$	$x + 2y = 3$
$x + 2(0) = 3$	$0 + 2y = 3$
$x = 3$	$2y = 3$
	$y = \frac{3}{2}$



The line passes through $(3, 0)$ and $(0, \frac{3}{2})$. Since the inequality has the symbol $>$, we draw a dashed line to connect the intercepts.

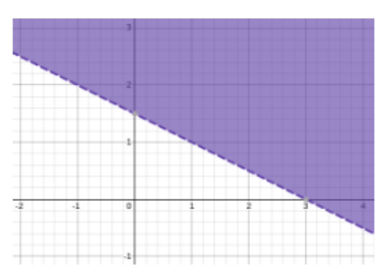
Step 2: Now, we choose a point on either side of the line graphed in step 1. Let's try the point $(0, 0)$:

$x + 2y > 3 \longrightarrow$ Given inequality

$0 + 2(0) > 3 \longrightarrow$ Test $(0, 0)$ by substituting 0 for x and 0 for y

$0 + 0 > 3 \longrightarrow$ Multiply

$0 > 3 \longrightarrow$ This is a false statement



Step 3: Since 0 is not greater than 3, we shade the side of the line (graphed in step 1) that does not contain the point $(0, 0)$.

We can also graph inequalities in the form $y > mx + b$ or $y < mx + b$ without using test points. The inequality symbols tells us which side of the line to shade:

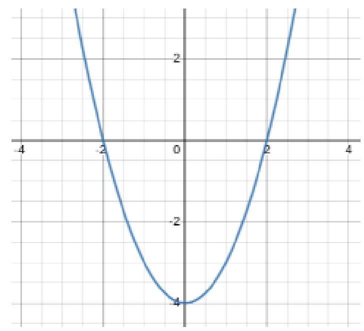
- If $y > mx + b$, shade the half-plane above the line $y = mx + b$
- If $y < mx + b$, shade the half-plane below the line $y = mx + b$

Example 2. Graphing a Nonlinear Inequality in Two Variables

Graph: $y \le x^2 - 4$

Step 1: We need to graph $y = x^2 - 4$. Let's find the x- and y-intercepts to graph the line.

Set $y = 0$ to find x-intercept	**Set $x = 0$ to find y-intercept**
$y = x^2 - 4$	$y = x^2 - 4$
$0 = x^2 - 4$	$y = (0)^2 - 4$
$4 = x^2$	$y = -4$
$\pm 2 = x$	



The line passes through $(2, 0)$, $(-2, 0)$ and $(0, -4)$. Since the inequality has the symbol $\le$, we draw the parabola with a solid line.

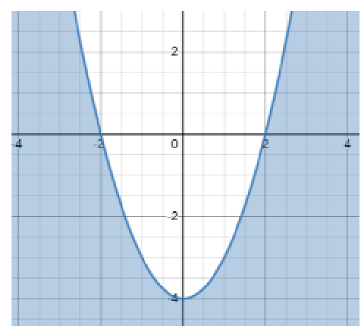
Step 2: Now, we choose a point inside or outside the parabola graphed in step 1. Let's try the point $(0, 0)$:

$y \le x^2 - 4 \quad \longrightarrow \quad$ Given inequality

$0 \le (0)^2 - 4 \quad \longrightarrow \quad$ Test $(0, 0)$ by substituting 0 for x and 0 for y

$0 \le -4 \quad \longrightarrow \quad$ This is a false statement

Step 3: Since 0 is not less than or equal to -4, we shade the outside of the parabola (graphed in step 1), which does not contain the point $(0, 0)$.



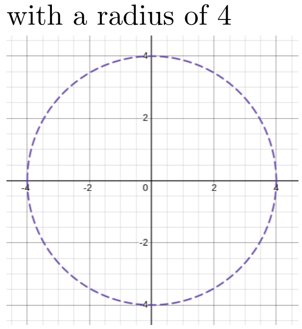
Example 3. Graphing a System of Inequalities

Graph the solution set of the system: $\begin{cases} x^2 + y^2 < 16 \\ x + y > 1 \end{cases}$

First, we need to graph both inequalities following the steps from the previous examples.

$x^2 + y^2 = 16$

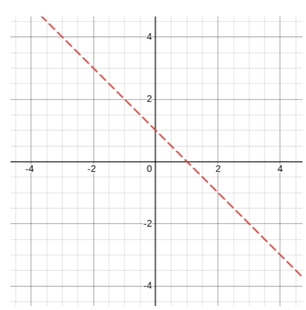
Recall: this is the equation of a circle centered at $(0, 0)$ with a radius of 4



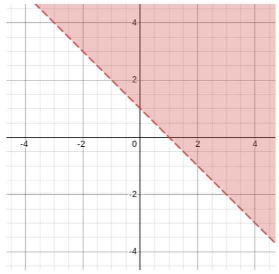
$x + y = 1$ Replace $>$ with $=$

$x = 1$ Let $y = 0$ to find x-intercept

$y = 1$ Let $x = 0$ to find the y-intercept



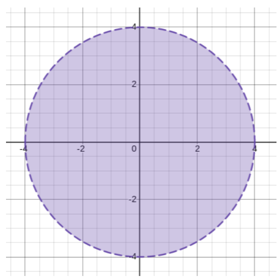
For the inequality $x + y > 1$, we can rewrite it as $y > -x + 1$. Then, use the rules from above to determine that we will shade above the line, like below:



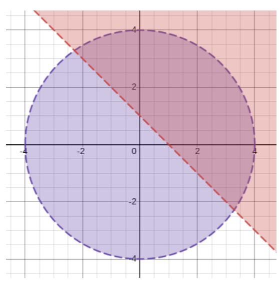
Now, we choose a test point to determine whether we shade the inside or outside of the circle. Let's test with the point $(0, 0)$:

$x^2 + y^2 < 16$	Given inequality
$0^2 + 0^2 < 16$	Test $(0, 0)$ by substituting 0 for x and 0 for y
$0 < 16$	This is a true statement

Therefore, we will shade the inside of the circle:



Putting the two graphs together, we get:

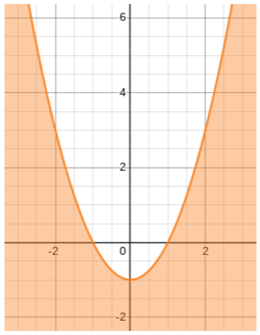


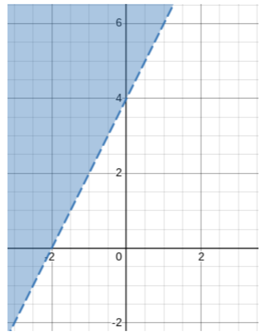
The solution set of the system of inequalities is the intersection of the two graphs (the area where both graphs are shaded), as shown above.

Practice Problems

1. Graph the inequality: $x^2 - y \geq 1$
2. Graph the inequality: $2x - y < -4$
3. Graph the solution set of the system: $\begin{cases} x^2 - y \geq 1 \\ 2x - y < -4 \end{cases}$

Solutions to Practice Problems

1. 

2. 

3.